

GRÖBNER—SHIRSHOV BASES FOR PRE-ASSOCIATIVE ALGEBRAS

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1. INTRODUCTION

The method of Gröbner bases developed by B. Buchberger in [7] is known to be one of the most effective techniques for solving various problems in different areas of mathematics wherever one needs to decide whether a given polynomial belongs to an ideal generated by a given family of polynomials (i.e., to solve the word problem). Here by “polynomial” we mean an element of the free associative and commutative algebra. Independently, A. I. Shirshov in [18] proposed similar technique for Lie algebras which may be easily reduced to the case of associative non-commutative algebras [2]. In contrast to the commutative Gröbner bases technique, the general one (Gröbner—Shirshov bases technique) does not provide an algorithm (in the classical “finite” sense) for solving the word problem, however, it is a powerful theoretic tool for studying a wide class of algebraic systems, see the reviews [5, 6] and references therein.

One of the most important applications of the Gröbner—Shirshov bases technique is the proof of PBW-like (Poincaré—Birkhoff—Witt) theorems for left adjointed functors to multiplication changing functors between varieties of linear algebras. Namely, suppose \mathcal{V} and \mathcal{W} are two (linear) operads governing the varieties of linear algebras called \mathcal{V} -algebras and \mathcal{W} -algebras, respectively. A morphism of operads $\omega : \mathcal{W} \rightarrow \mathcal{V}$ induces a functor from the variety of \mathcal{V} -algebras to the variety of \mathcal{W} -algebras. Such functors are called multiplication changing ones. Every multiplication changing functor has left adjointed functor which sends an arbitrary \mathcal{W} -algebra A to its universal enveloping \mathcal{V} -algebra $U_\omega(A)$. There exists canonical linear map $\iota : A \rightarrow U_\omega(A)$ which may not be injective, in general, such that $U_\omega(A)$ is generated by $\iota(A)$. Moreover, $U_\omega(A)$ carries a natural ascending filtration and its associated graded algebra $\text{gr } U_\omega(A)$ is also a \mathcal{V} -algebra. As it was proposed in [17], let us say the triple $(\mathcal{V}, \mathcal{W}, \omega)$ to have *PBW-property* if

$$\text{gr } U_\omega(A) \simeq U_\omega(A^{(0)})$$

as \mathcal{V} -algebras, where $A^{(0)}$ stands for the \mathcal{W} -algebra on same underlying space as A with trivial (zero) operations.

It was shown in [17] that many combinatorial properties of free \mathcal{V} - and \mathcal{W} -algebras are closely related provided that $(\mathcal{V}, \mathcal{W}, \omega)$ has the PBW-property. Gröbner—Shirshov bases technique is the traditional tool to decide whether $(\mathcal{V}, \mathcal{W}, \omega)$ has the PBW-property. To apply this technique, one has to describe the free \mathcal{V} -algebra, prove a

version of Composition-Diamond Lemma for \mathcal{V} -algebras, and find Gröbner—Shirshov basis (GSB) of $U_\omega(A)$ for a generic \mathcal{W} -algebra A .

Note that $U_\omega(A)$ has a standard presentation by generators and defining relations. Suppose the operad \mathcal{W} is generated by operations Ω , and $\nu(f)$ stands for the arity of an operation $f \in \Omega$. Let X be a linear basis of A , then $U_\omega(A)$ may be presented as $\mathcal{V}\langle X \rangle / I(\omega(f)(x_1, \dots, x_{\nu(f)}) - f(x_1, \dots, x_{\nu(f)}), x_i \in X, f \in \Omega)$, where $\mathcal{V}\langle X \rangle$ is the free \mathcal{V} -algebra generated by X and $I(S)$ denotes the ideal of $\mathcal{V}\langle X \rangle$ generated by S . The set S of defining relations stated here may not be a GSB, but if we can embed it into a GSB \bar{S} generating the same ideal $I(S) = I(\bar{S})$ then it remains to consider principal parts of the GSB \bar{S} obtained. If neither of these principal parts belong to X and the list of all these parts does not depend on the particular multiplication table of A then $(\mathcal{V}, \mathcal{W}, \omega)$ has the PBW-property [17].

Let us mention, for example, the paper [4] where the Gröbner—Shirshov bases method was developed for right-symmetric (pre-Lie) algebras known also as pre-Lie algebras. As an application, it was actually shown that $(\text{preLie}, \text{Lie}, (-))$ has the PBW-property. Here preLie is the operad governing the variety of algebras with one binary multiplication such that

$$(xy)z - x(yz) - (xz)y + x(zy) = 0,$$

Lie is the operad of Lie algebras, and $\omega = (-)$ turns xy into $xy - yx$. It turns out that, in these settings, the initial set of defining relations of the universal enveloping preLie -algebra of an arbitrary Lie algebra is a GSB.

In this paper, we consider similar problem in the associative settings and, as an application, prove that $(\text{Dend}, \text{As}, (*))$ has the PBW-property, where Dend is the operad governing the variety of dendriform algebras [15], i.e., algebras with two binary operations \prec and \succ satisfying the identities

$$\begin{aligned} x \succ (y \succ z) &= (x \succ y) \succ z + (x \prec y) \succ z, \\ x \succ (y \prec z) &= (x \succ y) \prec z, \\ x \prec (y \succ z) + x \prec (y \prec z) &= (x \prec y) \prec z, \end{aligned}$$

As is the operad of associative algebras, and $(*)$ turns xy into $x \succ y + x \prec y$.

We develop Gröbner—Shirshov bases method for dendriform algebras and, as an application, compute the GSB of the universal enveloping dendriform algebra of an arbitrary associative algebra. In contrast to the preLie case [4], multiplication table of A itself is not a GSB, but we still present the complete system of relations.

In fact, as we explain in Section 3, the relation between Lie and preLie is very similar to the relation between As and Dend , as well as between the operads governing Poisson and pre-Poisson algebras [1], Jordan and pre-Jordan algebras [13]. This is why we prefer the term “pre-associative algebra” instead of “dendriform algebra” for Dend -algebras.

2. GRÖBNER—SHIRSHOV BASES IN FREE ALGEBRAS

In this section, we recall the notion of a Gröbner—Shirshov basis in the free non-associative algebra $M\langle X \rangle$ and in the free associative algebra $\text{As}\langle U \rangle$. In this section, we will follow [2] and [3] with minor changes. We will also introduce some notations to be used later.

Let X be a nonempty set equipped with a well order \leq , and let X^{**} be the set of all (nonempty) non-associative words in the alphabet X . For $u \in X^{**}$, denote by $|u|$ the length of u . Define the *weight* $\text{wt}(u)$ of $u \in X^{**}$ as follows: for $u = x \in X$ put $\text{wt}(u) = (1, x)$, for $u = u_1 u_2$ put $\text{wt}(u) = (|u|, u_2, u_1)$. Extend the initial order on X to the order \leq on X^{**} by induction on the length:

$$u \leq v \iff \text{wt}(u) \leq \text{wt}(v) \text{ lexicographically.}$$

Obviously, this is a *monomial* order, i.e.,

$$u \leq v \Rightarrow wu \leq wv, \quad uv \leq vw$$

for all $u, v, w \in X^{**}$.

Remark 1. *This definition of order on X^{**} is slightly different from much more general one in [3], where the weight of $u = u_1 u_2$ is defined as $(|u|, u_1, u_2)$, but all statements remain valid.*

The set X^{**} is a linear basis of the free non-associative algebra $M\langle X \rangle$ generated by X . Suppose $0 \neq f, g \in M\langle X \rangle$, are non-associative polynomials, and let \bar{f}, \bar{g} be principal parts of f, g . Assume f and g are monic, i.e., the coefficients at their principal parts are equal to 1. If $\bar{f} = (u_1 \dots u_k \bar{g} v_1 \dots v_l)$ with some bracketing, $u_i, v_j \in X^{**}$, then

$$h = f - (u_1 \dots u_k g v_1 \dots v_l)$$

is called the *composition of inclusion* relative to $w = \bar{f}$.

A set S of monic non-associative polynomials is called a *Gröbner—Shirshov basis* in $M\langle X \rangle$ if for all $f, g \in S$ and for every their composition of inclusion h relative to a word w we have

$$h = \sum_i \alpha_i (u_{i1} \dots u_{ik_i} s_i v_{i1} \dots v_{il_i})_i,$$

where $\alpha_i \in \mathbb{k}$, $u_{ij}, v_{ij} \in X^{**}$ ($k_i, l_i \geq 0$), $s_i \in S$, $(\dots)_i$ denote some bracketings, and $(u_{i1} \dots u_{ik_i} \bar{s}_i v_{i1} \dots v_{il_i})_i < w$. This property of h is denoted $h \equiv 0 \pmod{S, w}$.

If a non-associative word $u \in X^{**}$ has no composition of inclusion with any of $g \in S$ then u is said to be an *S -reduced* word.

Theorem 1 ([3]). *Let S be a set of monic non-associative polynomials in $M\langle X \rangle$, and let $I(S)$ be the ideal of $M\langle X \rangle$ generated by S . Then the following statements are equivalent:*

- S is a Gröbner—Shirshov basis in $M\langle X \rangle$;
- $0 \neq f \in I(S)$ implies $\bar{f} = (u_1 \dots u_k s v_1 \dots v_l)$ for an appropriate $s \in S$;

- The set of all S -reduced words forms a linear basis of $M\langle X \mid S \rangle = M\langle X \rangle / I.(S)$.

Introduce the following relation on $M\langle X \rangle$:

$$f \rightarrow_S g \iff g = f - \alpha(u_1 \dots u_k s v_1 \dots v_l),$$

where $\bar{f} = (u_1 \dots u_k \bar{s} v_1 \dots v_l)$, $s \in S$, $u_i, v_i \in X^{**}$, and $\alpha \in \mathbb{k}$ is the coefficient at \bar{f} in f .

If there exists a chain $f \rightarrow_S g_1 \rightarrow_S \dots \rightarrow_S 0$ then we say that f may be reduced to zero modulo S . Denote it also by $f \rightarrow_S 0$.

Corollary 1 (Elimination of leading words). *Suppose S is a GSB in $M\langle X \rangle$, $f \in M\langle X \rangle$. Then $f \in I.(S)$ if and only if f may be reduced to zero modulo S .*

Now recall the associative case [2]. Given a nonempty set U , denote by U^* the set of all associative words in U relative to associative binary operation denoted by $*$, U^* is a linear basis of the free associative algebra $\text{As}\langle U \rangle$ generated by U . Denote by $|w|_*$ the length of a word $w \in U^*$, and let $U^\#$ be the set U^* together with empty word ϵ ; assume $|\epsilon|_* = 0$. Suppose U^* is equipped with a well monomial order relative to the operation $*$ (e.g., the deg-lex order based on a well order on the alphabet U).

Let $f, g \in \text{As}\langle U \rangle$ be monic polynomials. Define the following two types of compositions.

(AC1) If $\bar{f} = w_1 * \bar{g} * w_2$, $w_i \in U^\#$, then

$$h = f - w_1 * g * w_2$$

is called a *composition of inclusion* relative to $w = \bar{f}$.

(AC2) If $\bar{f} = u * w_1$, $\bar{g} = w_2 * u$, $u, w_1, w_2 \in U^*$, then

$$w_2 * f - g * w_1 \in \text{As}\langle U \rangle$$

is called a *composition of intersection* relative to $w = w_1 * \bar{g} = \bar{f} * w_2$.

A set $\Sigma \subset \text{As}\langle U \rangle$ of monic polynomials is a Gröbner—Shirshov basis in $\text{As}\langle U \rangle$ if for all $f, g \in \Sigma$ and for every their composition of inclusion h relative to a word w we have

$$h = \sum_i \alpha_i w_i * s_i * w'_i,$$

where $\alpha_i \in \mathbb{k}$, $w_i, w'_i \in U^\#$, $s_i \in \Sigma$, and $w_i * \bar{s}_i * \bar{w}'_i < w$. This property of h is denoted $h \equiv 0 \pmod{\Sigma, *, w}$.

If a word $u \in U^*$ has no composition of inclusion with $g \in \Sigma$ then u is said to be a Σ -reduced word.

Theorem 2 ([2]). *Let Σ be a set of monic non-associative polynomials in $\text{As}\langle U \rangle$, and let $I_*(\Sigma)$ be the ideal of $\text{As}\langle U \rangle$ generated by Σ . Then the following statements are equivalent:*

- Σ is a Gröbner—Shirshov basis in $\text{As}\langle U \rangle$;
- $f \in I_*(\Sigma)$ implies $\bar{f} = w * \bar{s} * w'$ for an appropriate $s \in \Sigma$, $w, w' \in U^\#$;

- The set all of Σ -reduced words forms a linear basis of $\text{As}\langle U \mid \Sigma \rangle = \text{As}\langle U \rangle / I_*(\Sigma)$.

In an obvious way, one may define what does it mean that $f \in \text{As}\langle U \rangle$ may be reduced to zero modulo Σ (c.f. Corollary 1). Denote it as $f \rightarrow_{\Sigma,*} 0$.

Corollary 2. *Suppose Σ is a GSB in $\text{As}\langle U \rangle$, $f \in \text{As}\langle U \rangle$. Then $f \in I_*(\Sigma)$ if and only if $f \rightarrow_{\Sigma,*} 0$.*

3. PRE-ALGEBRAS

Throughout this section, by an operad \mathcal{P} we mean a collection of linear \mathbb{k} -spaces $\mathcal{P}(n)$, $n \geq 1$, equipped with polylinear composition rule and linear action of the symmetric group S_n satisfying the well-known axioms of associativity, identity, and equivariance (see, e.g., [16]). For example, the class $\text{Vec}_{\mathbb{k}}$ of all linear \mathbb{k} -spaces with polylinear maps is a multi-category, but every single space $V \in \text{Vec}_{\mathbb{k}}$ may be considered as an operad by means of $V(n) = \text{Hom}_{\mathbb{k}}(V^{\otimes n}, V)$ with ordinary composition rule, where symmetric group acts by permutation of arguments.

Given an operad \mathcal{P} , a \mathcal{P} -algebra A is a morphism of operads (i.e., a functor on multi-categories) $\mathcal{P} \rightarrow \text{Vec}_{\mathbb{k}}$. Later we will often identify a morphism A and linear \mathbb{k} -space $A(\mathcal{P})$ considered as an operad in $\text{Vec}_{\mathbb{k}}$.

Example 1. Consider the family of \mathbb{k} -spaces $\text{Perm}(n) = \mathbb{k}^n$, $n \geq 1$, with standard bases $\{e_i^{(n)} \mid i = 1, \dots, n\}$. Define composition rule and symmetric group actions in a natural way ([8]):

$$e_i^{(n)}(e_{j_1}^{(m_1)}, \dots, e_{j_n}^{(m_n)}) = e_{m_1 + \dots + m_{i-1} + j_i}^{(m)}, \quad \sigma : e_i^{(n)} \mapsto e_{i\sigma}^{(n)}, \quad \sigma \in S_n.$$

The operad obtained is denoted by Perm . The class of Perm -algebras is a variety of all associative algebras satisfying the following identity:

$$(xy)z = (yx)z.$$

Recall the definition of what is a pre \mathcal{P} -algebra for a given operad \mathcal{P} [12].

Suppose $\Omega = \bigoplus_{n \geq 1} \Omega(n)$ is a graded linear \mathbb{k} -space, where each $\Omega(n)$ is an S_n -module.

Let $\mathcal{F} = \mathcal{F}_{\Omega}$ stand for the free operad generated by Ω .

Denote by $\Omega^{(2)}$ the following graded linear \mathbb{k} -space:

$$\Omega^{(2)} = \bigoplus_{n \geq 1} \Omega(n) \otimes \text{Perm}(n).$$

Extend the action of S_n by

$$(\omega \otimes e_i^{(n)})^{\sigma} = \omega^{\sigma} \otimes e_i^{(n)}, \quad \omega \in \Omega(n), \quad \sigma \in S_n,$$

and denote by $\mathcal{F}^{(2)} = \mathcal{F}_{\Omega^{(2)}}$ the free operad generated by $\Omega^{(2)}$.

Given an $\mathcal{F}^{(2)}$ -algebra A and a Perm -algebra P , define a morphism

$$P \boxtimes A : \mathcal{F} \rightarrow \text{Vec}_{\mathbb{k}}$$

as follows:

$$(P \boxtimes A)(\mathcal{F}) = P \otimes A \in \text{Vec}_{\mathbb{k}},$$

(1)

$$(P \boxtimes A)(\omega) : (p_1 \otimes a_1, \dots, p_n \otimes a_n) \mapsto \sum_{i=1}^n P(e_i^{(n)})(p_1, \dots, p_n) \otimes A(\omega \otimes e_i^{(n)})(a_1, \dots, a_n),$$

for $\omega \in \Omega(n)$, $p_i \in P$, $a_i \in A$, $i = 1, \dots, n$, $n \geq 1$.

Definition 1. Assume an operad \mathcal{P} is an image of \mathcal{F} , i.e., there exists an epimorphism $\tau : \mathcal{F} \rightarrow \mathcal{P}$.

An $\mathcal{F}^{(2)}$ -algebra A is called a pre \mathcal{P} -algebra if for every Perm-algebra P the morphism $P \boxtimes A$ is a \mathcal{P} -algebra (i.e., $\text{Ker } \tau \subseteq \text{Ker}(P \boxtimes A)$).

Obviously, it is enough to check this property for countably generated free Perm-algebra P only.

Let us consider one particular example in more details.

Example 2 (Pre-associative algebras). Suppose $\mathcal{P} = \text{As}$ is the operad of associative algebras, here $\Omega = \Omega(2)$, $\dim \Omega(2) = 2$, $\Omega(2) = \mathbb{k}\mu + \mathbb{k}\mu^{(12)}$, $(12) \in S_2$. For an As-algebra A , the image of μ is the binary operation of multiplication $A \otimes A \rightarrow A$, $\mu^{(12)}$ is the opposite multiplication.

Then $\Omega^{(2)}$ is a 4-dimensional space spanned by $\mu_i = \mu \otimes e_i^{(2)}$ and $\mu_i^{(12)} = \mu^{(12)} \otimes e_i^{(2)}$, $i = 1, 2$. In the usual notation, the images of μ_1 and μ_2 in an $\mathcal{F}^{(2)}$ -algebra correspond to two binary products denoted \succ and \prec , respectively.

According to the general Definition 1, an $\mathcal{F}^{(2)}$ -algebra A is a pre As-algebra if $P \boxtimes A$ is an associative algebra, i.e.,

$$(p_1 \otimes a_1)((p_2 \otimes a_2)(p_3 \otimes a_3)) = ((p_1 \otimes a_1)(p_2 \otimes a_2))(p_3 \otimes a_3)$$

for every $p_i \in P$, $a_i \in A$, where P is an arbitrary Perm-algebra.

The expansion (1) for $P \boxtimes A$ in this case turns into

$$(p \otimes a)(q \otimes b) = pq \otimes (a \succ b) + qp \otimes (a \prec b),$$

so $P \boxtimes A$ is associative if and only if

$$\begin{aligned} p_1 p_2 p_3 \otimes a_1 &\succ (a_2 \succ a_3) + p_1 p_3 p_2 \otimes a_1 \succ (a_2 \prec a_3) \\ &+ p_2 p_3 p_1 \otimes a_1 \prec (a_2 \succ a_3) + p_3 p_2 p_1 \otimes a_1 \prec (a_2 \prec a_3) \\ &= p_1 p_2 p_3 \otimes (a_1 \succ a_2) \succ a_3 + p_2 p_1 p_3 \otimes (a_1 \prec a_2) \succ a_3 \\ &+ p_3 p_1 p_2 \otimes (a_1 \succ a_2) \prec a_3 + p_3 p_2 p_1 \otimes (a_1 \prec a_2) \prec a_3. \end{aligned}$$

Collecting similar terms, we obtain the following necessary and sufficient conditions for A to be pre-associative:

$$(2) \quad \begin{aligned} a_1 \succ (a_2 \succ a_3) &= (a_1 \succ a_2) \succ a_3 + (a_1 \prec a_2) \succ a_3, \\ a_1 \succ (a_2 \prec a_3) &= (a_1 \succ a_2) \prec a_3, \\ a_1 \prec (a_2 \succ a_3) + a_1 \prec (a_2 \prec a_3) &= (a_1 \prec a_2) \prec a_3, \end{aligned}$$

i.e., A is a dendriform algebra ($A \in \text{Dend}$) in the sense of [15].

Example 3 (Pre-Lie algebras). Let $\mathcal{P} = \text{Lie}$ be the operad of Lie algebras. Then $\Omega = \Omega(2)$, $\dim \Omega(2) = 1$, $\Omega(2) = \mathbb{k}\nu$, $\nu^{(12)} = -\nu$. The class of pre Lie-algebras determined by Definition 1 coincides with the variety of left-symmetric algebras (LSAs), also called pre-Lie algebras [10, 14, 20].

Remark 2. The operad $\text{pre } \mathcal{P}$ governing the variety of all $\text{pre } \mathcal{P}$ -algebras coincides with the Manin black product $\text{pre Lie} \bullet \mathcal{P}$ [19].

The structure of a pre-associative algebra may be equivalently described by means of two operations

$$(3) \quad x * y = x \succ y + x \prec y \quad \text{and} \quad xy = x \succ y.$$

Let us rewrite axioms (2) in terms of these operations:

$$(4) \quad \begin{aligned} a_1 * (a_2 * a_3) &= (a_1 * a_2) * a_3, \\ (a_1 * a_2)a_3 &= a_1(a_2a_3), \\ a_1(a_2 * a_3) &= (a_1a_2) * a_3 - (a_1, a_2, a_3), \end{aligned}$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator. For example, (4) imply the following relation in pre As :

$$(5) \quad a_1(a_2 * a_3 * a_4) = (a_1a_2) * a_3 * a_4 - (a_1, a_2, a_3) * a_4 + (a_1, a_2, a_3)a_4 - (a_1a_2)(a_3a_4).$$

It is easy to see that the identities (4) allow to rewrite every term in a pre-associative algebra generated by a set X as a linear combination of the following monomials:

$$(6) \quad u_1 * u_2 * \cdots * u_k, \quad k \geq 1, \quad u_i \in X^{**},$$

where X^{**} denotes, as above, the set of all (nonempty) non-associative words in the alphabet X relative to the second operation in (3).

Theorem 3. The set of all monomials (6) is a linear basis of the free pre-associative algebra $\text{pre As}\langle X \rangle$ generated by X . Therefore, $\text{pre As}\langle X \rangle$ is isomorphic to $T_0(M\langle X \rangle)$ as a linear space, where $M\langle X \rangle$ stands for the free non-associative algebra and T_0 denotes tensor algebra without identity.

Proof. It is enough to consider the case $|X| = 1$ since the operad pre As is nonsymmetric. One may easily calculate the number of different monomials (6) of length n is equal to the n th Catalan number C_n which is known to be the dimension of $\text{Dend}(n)$ [15]. Hence, the words (6) are linearly independent in $\text{Dend}\langle X \rangle = \text{pre As}\langle X \rangle$. \square

Lemma 1. *Let S be a subset of $\text{pre As}\langle X \rangle$. Denote by $I(S)$ the ideal of the free pre-associative algebra generated by S . Then $I(S)$ coincides with the ideal $I_*(\tilde{S})$ generated in the free associative algebra $\text{As}\langle U \rangle$, $U = X^{**}$, by the set \tilde{S} of all elements*

$$(u_1 \dots u_l s v_1 \dots v_r), \quad u_i, v_j \in U, \quad l, r \geq 0, \quad s \in S$$

with all possible bracketings.

Proof. Obviously, $I_*(\tilde{S}) \subseteq I(S)$. Conversely, $I_*(\tilde{S})$ is an ideal in $\text{pre As}\langle X \rangle$ due to (4). Hence, $I_*(\tilde{S}) \supseteq I(S)$. \square

4. GRÖBNER—SHIRSHOV BASES IN PRE-ASSOCIATIVE ALGEBRAS

Let X be a nonempty set. Denote by U the set X^{**} of all non-associative words in the alphabet X . By U^* we denote the set of all associative words in U relative to binary operation $*$. Theorem 3 implies U^* to be a linear basis of $\text{pre As}\langle X \rangle$. Moreover, $\text{pre As}\langle X \rangle$ as an associative algebra with respect to $*$ is isomorphic to $\text{As}\langle U \rangle$.

Assume X is well-ordered, and this order is extended to U as in Section 2. Then extend the order to U^* in a monomial way such that $|u|_* > 1$ and $|v|_* = 1$ imply $u > v$ for $u, v \in U^*$ (e.g., the standard deg-lex order has this property). The latter condition guarantees that a polynomial $f \in \text{pre As}\langle X \rangle$ is $*$ -free (i.e., belongs to $\mathbb{k}U$) if so is its principal part.

Consider the following types of compositions in pre-associative algebras.

- (C1) The composition of $*$ -inclusion for $f, g \in \text{pre As}\langle X \rangle$ is defined in the same way as the composition (AC1) in the free associative algebra $\text{As}\langle U \rangle$;
- (C2) The composition of $*$ -intersection coincides with (AC2) in $\text{As}\langle U \rangle$;
- (C3) Suppose $f, g \in \text{pre As}\langle X \rangle$, $|\bar{g}|_* = 1$. Assume $\bar{f} = v * u * v'$, $v, v' \in U^\#$, where $u \in \widehat{\{g\}}$, i.e.,

$$u = (w_1 \dots w_k \bar{g} w'_1 \dots w'_{k'}) \in U$$

with respect to some bracketing (\dots) , $w_i, w'_i \in U$, $k + k' > 0$. Then

$$h = f - v * (w_1 \dots w_k g w'_1 \dots w'_{k'}) * v'$$

is called the composition of \succ -inclusion relative to $w = \bar{f}$;

- (C4) Let $f \in \text{pre As}\langle X \rangle$, $|\bar{f}|_* > 1$. Then for every $v \in U$ the element $h = f v \in \text{pre As}\langle X \rangle$ is called the composition of *right multiplication* relative to $w = \bar{f}v$;
- (C5) Let $f \in \text{pre As}\langle X \rangle$, $|\bar{f}|_* > 1$. Then for every $u \in U^*$ the element $h = u f \in \text{pre As}\langle X \rangle$ is called the composition of *left multiplication* relative to $w = \overline{u f}$.

Suppose $S \subseteq \text{pre As}\langle X \rangle$ is a set of polynomials. Denote by $I(S)$ the ideal of $\text{pre As}\langle X \rangle$ generated by S . Let us split S as $S = S_0 \dot{\cup} S_1$, where $S_0 = \{f \in S \mid |\bar{f}|_* = 1\}$, i.e., S_0 consists of all $*$ -free polynomials, $S_0 = \mathbb{k}U \cap S$, $S_1 = S \setminus S_0$. Define

$$\hat{S}_0 = \bigcup_{k \geq 0} S_0^{(k)}, \quad S_0^{(0)} = S_0, \quad S_0^{(k+1)} = \{v f, f v \mid f \in S_0^{(k)}, v \in U\}.$$

Denote

$$\hat{S} = \hat{S}_0 \cup S_1.$$

Note that $I_*(\hat{S}) \subseteq I(S)$ since the latter is equal to $I_*(\tilde{S})$ by Lemma 1.

A composition h of type (C1)–(C3) relative to a word w is said to be *trivial modulo* S if $h \equiv 0 \pmod{\hat{S}, *, w}$. Similarly, a composition h of type (C4)–(C5) relative to w is trivial modulo S if

$$h = \sum_i \alpha_i w_i * s_i * w'_i,$$

where $\alpha_i \in \mathbb{k}$, $w_i, w'_i \in U^\#$, $s_i \in \hat{S}$, and $w_i * \bar{s}_i * \bar{w}'_i \leq w$. Note that for a composition of type (C4) we should have $w_i = w'_i = \epsilon$ and $s_i \in \hat{S}_0$ since $|\bar{h}|_* = 1$.

Definition 2. A set of monic pre-associative polynomials $S \subseteq \text{pre As}\langle X \rangle$ is a Gröbner—Shirshov basis (GSB) in $\text{pre As}\langle X \rangle$ if for all $f, g \in S$ every composition of type (C1)–(C5) is trivial modulo S .

Assume S is a set of monic pre-associative polynomials.

Lemma 2. *The set \hat{S} is a GSB in $\text{As}\langle U \rangle$ if and only if for all $f, g \in S$ every their composition of type (C1)–(C3) is trivial modulo S .*

Proof. If \hat{S} is a GSB in $\text{As}\langle U \rangle$ then compositions (C1)–(C3) of elements of S coincide with compositions (AC1) or (AC2).

Conversely, since all compositions of type (C3) are trivial, S_0 is a GSB in $M\langle X \rangle = \mathbb{k}U$.

Consider $f, g \in \hat{S}$ as elements of $\text{As}\langle U \rangle$, suppose they have a composition h of type (AC1) or (AC2) relative to a word $w \in U^*$. If h is a composition of intersection (AC2) then both f, g belong to S_1 and $h \equiv 0 \pmod{\hat{S}, *, w}$ by definition. Assume h is a composition of inclusion (AC1). If $f \in \hat{S}_0, g \in S_1$ (or converse) then h coincides with a composition of type (C3) and thus $h \rightarrow_{\hat{S}} 0$. Consider the case when both f, g belong to \hat{S}_0 , and h is a composition of type (AC1) relative to w . Then

$$\bar{f} = \bar{g} = w, \quad h = f - g \in I(S_0),$$

where either $h = 0$ or $\bar{h} < w$. By Corollary 1, $h \rightarrow_{S_0} 0$, i.e., $h \equiv 0 \pmod{S_0, w}$. The latter trivially implies $h \equiv 0 \pmod{\hat{S}, *, w}$. \square

Lemma 3. *If for all $f \in S$ every composition of type (C4), (C5) is trivial modulo S then $I_*(\hat{S}) = I(S)$.*

Proof. Obviously, $I_*(\hat{S}) \subseteq I(S)$ for an arbitrary S . To prove the converse, it is enough to show that the set $I_*(\hat{S})$ is an ideal of $\text{pre As}\langle X \rangle$.

It follows from (4) that for every $s \in \hat{S}_0, w = v_1 * \cdots * v_k \in U^*, v_i \in U$, we have $sw, ws \in I_*(\hat{S}_0)$, moreover, $ws \in \mathbb{k}\hat{S}_0$.

Suppose b is a generic element of $I_*(\hat{S})$,

$$b = \sum_i \alpha_i w_i * s_{0i} * w'_i + \sum_j \beta_j v_j * s_{1j} * v'_j,$$

where $w_i, w'_i, v_j, v'_j \in U^\#$, $s_{0i} \in \hat{S}_0$, $s_{1j} \in S_1$, $\alpha_i, \beta_j \in \mathbb{k}$.

It is easy to derive from (5) that $w(w_i * s_{0i} * w'_i) \in I_*(\hat{S}_0)$ for all $w \in U^*$ since $(w, w_i, s_{0i}), (ww_i)(s_{0i}w'_i) \in \hat{S}_0$. Consider the second group of summands:

$$w(v_j * s_{1j} * v'_j) = (wv_j) * s_{1j} * v'_j - (w, v_j, s_{1j}) * v'_j + (w, v_j, s_{1j})v'_j - (wv_j)(s_{1j}v'_j) \in I_*(\hat{S})$$

provided that $(w, v_j, s_{1j}) \in I_*(\hat{S})$. The latter holds since $(wv_j)s_{1j}$ and $w(v_js_{1j}) = (w * v_j)s_{1j}$ are compositions of type (C5).

Therefore, $wb \in I_*(\hat{S})$. In a similar way, one may show $bw \in I_*(\hat{S})$ for all $w \in U^*$. This implies $I_*(\hat{S})$ to be an ideal of pre-associative algebra $\text{pre As}\langle X \rangle$. \square

Remark 3. If \hat{S} is a GSB in $\text{As}\langle U \rangle$ then the converse holds: $I_*(\hat{S}) = I(S)$ implies that all compositions of type (C4), (C5) of elements of S are trivial.

Indeed, if $f \in S$ then $wf, fu \in I(S) = I_*(\hat{S})$ for all $w \in U^*$, $u \in U$. The latter means, in particular, $wf, fu \rightarrow_{\hat{S},*} 0$ in $\text{As}\langle U \rangle$, i.e., compositions (C4), (C5) are trivial modulo S in $\text{pre As}\langle X \rangle$.

Theorem 4. Suppose $S \subseteq \text{pre As}\langle X \rangle$ is a set of monic pre-associative polynomials, and $I(S)$ is the ideal in $\text{pre As}\langle X \rangle$ generated by S . Then the following statements are equivalent:

- S is a GSB in $\text{pre As}\langle X \rangle$;
- $f \in I(S)$ implies $\bar{f} = w * \bar{s} * w'$ for an appropriate $s \in \hat{S}$, $w, w' \in U^\#$;
- The set all of \hat{S} -reduced words in $\text{As}\langle U \rangle$ is a linear basis of $\text{pre As}\langle X \mid S \rangle = \text{pre As}\langle X \rangle / I(S)$.

Proof. (i) \Rightarrow (ii)

If S is a GSB then $f \in I(S)$ implies $f \in I_*(\hat{S})$ by Lemma 3. As \hat{S} is a GSB in $\text{As}\langle U \rangle$ by Lemma 2, \bar{f} contains a subword \bar{s} for an appropriate $s \in \hat{S}$.

(ii) \Rightarrow (iii)

Obviously, every element of $\text{pre As}\langle X \mid S \rangle$ may be rewritten as a combination of \hat{S} -reduced words from U^* . If (ii) holds, these words are linearly independent.

(iii) \Rightarrow (i)

By Lemma 1, $\text{pre As}\langle X \mid S \rangle$ is isomorphic to $\text{As}\langle U \mid \tilde{S} \rangle$ as a linear space. As $\hat{S} \subseteq \tilde{S}$, there is a linear map $\alpha : \text{As}\langle U \mid \hat{S} \rangle \rightarrow \text{pre As}\langle X \mid S \rangle$ such that the diagram

$$\begin{array}{ccc} \text{pre As}\langle X \rangle & \xrightarrow{\cong} & \text{As}\langle U \rangle \\ \downarrow & & \downarrow \\ \text{pre As}\langle X \mid S \rangle & \xleftarrow{\alpha} & \text{As}\langle U \mid \hat{S} \rangle \end{array}$$

commutes.

In general, the set of all \hat{S} -reduced words is a complete (but not necessarily linearly independent) set in the space $\text{As}\langle U \mid \hat{S} \rangle$. In the case when (iii) holds, the images of these words form a basis in $\text{pre As}\langle X \mid S \rangle$, so the condition (iii) of Theorem 2 holds for \hat{S} . Hence, S satisfies the conditions of Lemma 2. Moreover, α has to be a linear isomorphism, so $I_*(\hat{S}) = I_*(\tilde{S}) = I(S)$ by Lemma 1. Thus, Lemma 3 and Remark 3 imply S to be a GSB in $\text{pre As}\langle X \rangle$. \square

Therefore, in order to check whether S is a GSB in $\text{pre As}\langle X \rangle$, one has to compute all compositions of type (C1)–(C3) of all elements of S first and show they are all trivial. Next, apply elimination of leading monomial of \hat{S} to all compositions of type (C4), (C5) to show they all reduce to zero.

5. APPLICATIONS

Suppose $A \in \text{pre As}$ is defined as a quotient of $\text{pre As}\langle X \rangle$ relative to an ideal I . If $I = I(S)$ for a GSB S then, by abuse of terminology, we say S is a GSB of the pre-associative algebra A .

In this section, we compute Gröbner—Shirshov bases for two series of pre-associative algebras: for free Zinbiel algebras and for universal pre-associative envelopes of associative algebras.

5.1. GSB of the free pre-commutative algebra. Recall that Zinbiel algebra is the name proposed by J.-M. Lemaire (see [15]) for commutative dendriform algebras, i.e., for pre-commutative algebras, in our terminology.

Namely, Definition 1 applied to the operad Com governing the variety of associative and commutative algebras shows that pre Com is defined by (2) together with $a_1 \succ a_2 = a_2 \prec a_1$. Therefore, pre-commutative algebras may be considered as systems with one operation $a_1 a_2 = a_1 \succ a_2$ satisfying

$$(7) \quad (a_1 a_2 + a_2 a_1) a_3 = a_1 (a_2 a_3).$$

We will use the following notation: for $a_1, a_2, \dots, a_n \in A$, $A \in \text{pre As}$, let $[a_1 \dots a_n]$ denote $(\dots((a_1 a_2) a_3) \dots a_n)$.

Theorem 5. *The following polynomials in $\text{pre As}\langle X \rangle$ form a GSB of the free pre-commutative algebra $\text{pre Com}\langle X \rangle$:*

$$(8) \quad M_{u,v} = u * v - uv - vu,$$

$$(9) \quad Z_{u,v,w} = u(vw) - (uv)w - (vu)w,$$

where $u, v, w \in U$.

Proof. Denote the set of polynomials in the statement by S .

Obviously, the compositions of $*$ -inclusion (C1) are the compositions of $M_{u,v(wp)}$ and $Z_{v,w,p}$ (relative to $u * v(wp)$) or $M_{u(vw),p}$ and $Z_{u,v,w}$ (relative to $u(vw) * p$). For example, the first of these compositions is equal to

$$h = u(v(wp)) + (v(wp))u - u * ((vw)p) - u * ((wv)p) \rightarrow_S Z_{v,w,p}u + uZ_{v,w,p} \rightarrow_{\hat{S}} 0,$$

so $h \equiv 0 \pmod{\hat{S}, *, u * (v(wp))}$. The second one is trivial by similar reasons.

Let us compute a composition of $*$ -intersection (C2): the composition of $M_{u,v}$ and $M_{v,w}$ (relative to $u * v * w$) is equal to

$$h = (uv + vu) * w - u * (vw + wv).$$

By definition, $h \rightarrow_{S,*} f = w(uv) + (uv)w + w(vu) + (vu)w - u(vw) - u(wv) - (vw)u - (wv)u$. The latter may be presented as

$$f = Z_{w,u,v} + Z_{w,v,u} - Z_{u,v,w} - Z_{u,w,v},$$

so $h \equiv 0 \pmod{\hat{S}, *, u * v * w}$.

Consider compositions of type (C3). Suppose $s_1 \in S$ and $s_2 \in S_0$ have a composition of \succ -inclusion. If $s_1 \in S_0$ then one may apply Theorem 1 and the known result on the linear basis of the free Zinbiel algebra [15] to conclude that the set of $\{Z_{u,v,w} \mid u, v, w \in X^{**}\}$ is a GSB in $M\langle X \rangle$, so it is closed with respect to all compositions of inclusion. However, it is not hard to compute all possible compositions straightforwardly and show they are all trivial.

If $s_1 \in S_1$ then $s_1 = M_{v,w}$ for $v, w \in U$. Assume \hat{s}_2 is a subword in v . Then the composition of \succ -inclusion is equal to

$$h = vw + wv - (v_1 + v_2) * w,$$

where $v - v_1 - v_2 \in \hat{S}$, $v > v_1, v_2$. Obviously,

$$h \rightarrow_{S,*} vw + wv - v_1w - wv_1 - v_2w - wv_2 \equiv 0 \pmod{\hat{S}, *, v * w}$$

The second case, when \bar{s}_2 is a subword in w , is completely analogous.

Therefore, by Lemma 2 \hat{S} is a GSB in $\text{As}\langle U \rangle$. It remains to show that all compositions (C4) and (C5) of right and left multiplication belong to $I_*(\hat{S})$.

Relation (8) shows that a composition of right multiplication of $M_{u,v}$ by w is exactly $Z_{u,v,w}$. For compositions of left multiplication, note that (4) implies

$$(10) \quad (w_1 * \dots * w_k)(u * v) = [w_1, \dots, w_k, u] * v - [w_1, \dots, w_k, u]v + [w_1, \dots, w_k, u, v],$$

in $\text{pre As}\langle X \rangle$, where $[x_1, \dots, x_m] = x_1(x_2 \dots (x_{m-1}x_m) \dots)$. Therefore, the composition of left multiplication of M_{u*v} by $w = w_1 * \dots * w_k$ is equal to

$$h = [w_1, \dots, w_k, u] * v - [w_1, \dots, w_k, u]v - [w_1, \dots, w_k, v, u].$$

Obviously, $h \rightarrow_{S,*} f = [v, w_1, \dots, w_k, u] - [w_1, \dots, w_k, v, u]$, where $f \in I_*(\hat{S})$: for $k = 1$

$$f = [v, w_1, u] - [w_1, v, u] = Z_{v,w_1,u} - Z_{w_1,v,u},$$

and

$$f = Z_{v,w_1,[w_2,\dots,w_k,u]} - Z_{w_1,v,[w_2,\dots,w_k,u]} + w_1([v, w_2, \dots, w_k, u] - [w_2, \dots, w_k, u]) \in I_*(\hat{S})$$

for $k > 1$ by induction.

Hence, S is a GSB in $\text{pre As}\langle X \rangle$, and monomials of the form $(\dots(x_1x_2)\dots x_n)$, $x_i \in X$, form a linear basis of $D = \text{pre As}\langle X \mid S \rangle$. Since all these monomials belong to U , the entire D satisfies pre-commutativity (Zinbiel) identity, so $D \simeq \text{pre Com}\langle X \rangle$. \square

5.2. GSB of the universal enveloping pre-associative algebra of an associative algebra. Every pre-associative algebra P considered as a space with one product $x * y = x \prec y + x \succ y$ is an ordinary associative algebra denoted $P^{(+)}$. For every associative algebra A with a product denoted by \cdot there exists its universal enveloping pre-associative algebra $U_*(A)$. Namely, A maps into $U_*(A)^{(\cdot)}$, $U_*(A)$ is generated (as a pre-algebra) by the image of A , and every homomorphism $A \rightarrow P^{(\cdot)}$ of ordinary algebras may be extended to a homomorphism $U_*(A) \rightarrow P$ of pre-algebras. It is clear that $U_*(A)$ may be presented by generators and relations as follows. Suppose X is a linear basis of A , then

$$U_*(A) \simeq \text{pre As}\langle X \mid x * y - x \cdot y, x, y \in X \rangle.$$

Introduce the following notation: $L_{w_1,\dots,w_k}(u) = [w_1, \dots, w_k, u]$ for $w_1, \dots, w_k, u \in U$, $k \geq 0$ (for $k = 0$, denote the operator L_{w_1,\dots,w_k} by $\mathbf{1}$). It follows from (10) that for every $a, b \in X$ and for every $L = L_{w_1,\dots,w_k}$

$$M_{L,a,b} = L(a) * b - L(a)b + L(ab) - L(a \cdot b)$$

belongs to the ideal of $\text{pre As}\langle X \rangle$ generated by $x * y - x \cdot y$, $x, y \in X$. Note that an arbitrary word in U may be uniquely presented as $L(a)$. Moreover,

$$W_{L,a,b,u} = M_{L,a,b}u = L(a)(bu) - (L(a)b)u + L(ab)u - L(a \cdot b)u$$

also belongs to the same ideal.

Theorem 6. *The set of relations $M_{L,a,b}$, $W_{L,a,b,u}$, where $L = L_{w_1,\dots,w_k}$, $w_1, \dots, w_k \in U$ ($k \geq 0$), $a, b \in X$, $u \in U$, is a GSB in $\text{pre As}\langle X \rangle$.*

This is a GSB of the universal enveloping pre-associative algebra $U_*(A)$ of an associative algebra A .

Proof. Denote by S the set of relations in the statement. Let us compute all compositions (C1)–(C3) of S to make sure \hat{S} is a GSB in $\text{As}\langle U \rangle$.

The only composition of $*$ -intersection (C1) comes from $M_{1,a,b}$ and $M_{1,b,c}$, $a, b, c \in X$. It is trivial due to associativity of A .

Let us consider compositions (C2) and (C3) together since (C2) is a “degenerate” case of (C3). Suppose $M_{L,a,b}$ and $W_{L',c,d,u}$ have a composition of $*$ -inclusion or \succ -inclusion, $L = L_{w_1,\dots,w_k}$. There are two possible cases:

- (1) $L'(c)(du)$ is a subword in a word w_i ;

- (2) $L'(c)(du)$ coincides with $L_{w_l, \dots, w_k}(a)$, $l \geq 1$ ($l = 1$ corresponds to composition of $*$ -inclusion).

In the first case, there exist $w_{i1}, w_{i2}, w_{i3} \in U$ such that $w_i - w_{i1} - w_{i2} - w_{i3} \in \hat{S}_0$, $w_{ij} < w_i$. Then for $L_j = L_{w_1, \dots, w_{ij}, \dots, w_k}$, $j = 1, 2, 3$, we have $L(w) - L_1(w) - L_2(w) - L_3(w) \in \hat{S}_0$, $L_j(w) < L(w)$ for all $w \in U$. The composition of \succ -inclusion has the form

$$h = L(a)b - L(ab) + L(a \cdot b) - \sum_{j=1}^3 L_j(a) * b,$$

and, obviously, $h \equiv 0 \pmod{\hat{S}, *, L(a) * b}$.

In the second case, $L(a) = [w_1, \dots, w_{l-1}, L'(c)(du)]$, so $u = L''(a)$, where $L'' = L_{v_1, \dots, v_m}$ for some $v_1, \dots, v_m \in U$, and $L(x) = [w_1, \dots, w_{l-1}, L'(c)(dL''(x))]$ for $x \in U$. Denote $L_{(1)} = L_{w_1, \dots, w_{l-1}}$. Then the composition of \succ -inclusion (or $*$ -inclusion for $l = 1$) is equal to

$$\begin{aligned} h &= L(a)b - L(ab) + L(a \cdot b) - L_{(1)}((L'(c)d)L''(a) - L'(cd)L''(a) + L'(c \cdot d)L''(a)) * b \\ &= L(a)b - L(ab) + L(a \cdot b) - L_{(11)}(a) * b + L_{(12)}(a) * b - L_{(13)}(a) * b \\ &\equiv L(a)b - L(ab) + L(a \cdot b) - L_{(11)}(a)b + L_{(11)}(ab) - L_{(11)}(a \cdot b) \\ &+ L_{(12)}(a)b - L_{(12)}(ab) + L_{(12)}(a \cdot b) - L_{(13)}(a)b + L_{(13)}(ab) - L_{(13)}(a \cdot b) \pmod{S, *L(a)*b}, \end{aligned}$$

where

$$\begin{aligned} L_{(11)} &= L_{w_1, \dots, w_{l-1}, L'(c)d, v_1, \dots, v_m}, \\ L_{(12)} &= L_{w_1, \dots, w_{l-1}, L'(cd), v_1, \dots, v_m}, \\ L_{(13)} &= L_{w_1, \dots, w_{l-1}, L'(c \cdot d), v_1, \dots, v_m}. \end{aligned}$$

Note that

$$L(w) = L_{(1)}(L'(c)(dL''(w))) \equiv L_{(11)}(w) - L_{(12)}(w) + L_{(13)}(w) \pmod{\hat{S}, *, L(a) * b}$$

for every $w \in U$. Hence,

$$h \equiv 0 \pmod{\hat{S}, *, L(a) * b}.$$

We have shown \hat{S} to be a GSB in $\text{As}\langle U \rangle$. To prove the theorem, it remains to check that all compositions of type (C4), (C5) of relations from S are trivial modulo \hat{S} .

Obviously, the composition of right multiplication of $M_{L,a,b}$ by $u \in U$ is equal to $W_{L,a,b,u}$, i.e., it is trivial. Moreover, $uM_{L,a,b} = M_{L',a,b}$, where $L'(x) = uL(x)$, $u \in U$. Therefore, S is a GSB in $\text{pre As}\langle X \rangle$. \square

Corollary 3. *Linear basis of the universal enveloping pre-associative algebra $U(A)$ of an associative algebra A consists of $u_1 * u_2 * \dots * u_n$, $n \geq 1$, $u_i \in U$, where $|u_2|, \dots, |u_n| > 1$ and neither of u_i contains a subword of the form $w(xv)$, $w, v \in U$, $x \in X$. In particular, the pair of varieties As , pre As has PBW-property in the sense of [17].*

5.3. Open problems. One of the most intriguing problems related with application of GSB theory for pre-associative algebras is related with the morphism of operads

$$(-) : \text{pre Lie} \rightarrow \text{pre As}$$

defined by $xy \mapsto x \succ y - y \prec x$. Indeed, every pre-associative algebra A with respect to new operation $a \cdot b = a \succ b - b \prec a$, $a, b \in A$, is a pre-Lie algebra $A^{(-)}$.

Problem 1. Find the Gröbner—Shirshov basis of the universal pre-associative envelope

$$U_{(-)}(L) = \text{pre As}\langle X \mid a * b - ab - ba + b \cdot a, a, b \in X \rangle, \quad X \text{ is a basis of } L,$$

of a pre-Lie algebra L .

Problem 2. Whether the triple $(\text{pre As}, \text{pre Lie}, (-))$ has the PBW-property?

There is a morphism ψ from the operad Brace governing the class of brace-algebras [11] to the operad pre As described in [9]. It was shown in [9] that the triple $(\text{pre As}, \text{Brace}, \psi)$ has the PBW-property.

Problem 3. Find the GSB of $U_{\psi}(B)$ for a brace-algebra B .

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